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Nonlinear parabolic problems in unbounded domains

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Synopsis

We show the existence of weak solutions of nonlinear parabolic partial differential equations in unbounded domains, provided that a variant of the Leray–Lions conditions is satisfied.

1. Introduction

In this paper we consider parabolic problems of the form

$$(P)\begin{cases} \dfrac{\partial u}{\partial t}+\mathscr{A}u=f & \text{in} \quad Q=\Omega\times(0,T) \\ u=0 & \text{on} \quad \Sigma=\partial\Omega\times(0,T) \\ u(0)=u_0 & \text{in} \quad \Omega, \end{cases}$$

where Ω denotes a (possibly unbounded) domain in $\mathbb{R}^N(N\geqq 1)$ and $T>0$. The operator $\mathscr{A}$ is given by

$$(\mathscr{A}u)(x,t)=\sum_{|\alpha|\leqq m}(-1)^{|\alpha|}D^\alpha A_\alpha(x,t,\delta u(x,t),D^m u(x,t)) \quad (x,t)\in Q.$$

[We follow the notation of **8**]. The real functions A_α satisfy a variant of the Leray–Lions conditions. Corresponding elliptic problems in unbounded domains have been studied successfully by the authors of [**2–5**]. They overcame the difficulties occurring when the Sobolev embedding theorem is not valid in unbounded domains, by finding a substitute for these compact embeddings. They showed that multiplication by certain functions induces a compact mapping of some (weighted) Sobolev space to appropriate L^q-spaces. A much simpler approach that does not require any special knowledge of Sobolev spaces over unbounded domains has been proposed in [**6**]. This method will be applied here to get weak solutions of the initial boundary value problem. Compared to the elliptic case new difficulties arise when one has to define the derivative of a function u with respect to t. We will overcome them by using a substitution which leads to an equivalent problem in a reflexive Banach space $\mathscr{W}$ with $\mathscr{W}\subset L^2(\Omega\times(0,T))\subset\mathscr{W}'$. The distribution derivative of $u\in\mathscr{W}$ with respect to t can then be defined appropriately.

2. Definitions and Statement of the Result

Let Ω be a domain in $\mathbb{R}^N$ with sufficiently smooth boundary $\partial\Omega$. We assume that we have a representation $\Omega = \bigcup_{k=1}^{\infty} \omega_k$, where $\{\omega_k\}$ is an increasing sequence of bounded subdomains of Ω with smooth boundary. To each $k \in \mathbb{N}$ we suppose that there exists a function $\varphi_k \in C_0^\infty(\mathbb{R}^N)$ with values in $[0, 1]$, such that $\varphi_k(x) = 1\ \forall x \in \omega_k$ and $\operatorname{supp} \varphi_k \subset \omega_{k+1}$.

By N_1 and N_2 we denote the number of derivatives in x of the order $\leqq m-1$ and of the order m, respectively. The following conditions of Leray–Lions type are imposed on the functions $A_\alpha(|\alpha| \leqq m)$:

(A1) Each $A_\alpha: Q \times \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \to \mathbb{R}$ satisfies the Carathéodory conditions, i.e. $A_\alpha(x, t, \eta, \xi)$ is measurable in $(x, t) \in Q$ for all fixed $(\eta, \xi) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and continuous in (η, ξ) for almost all fixed (x, t).

There exist constants $q: 1 < q < \infty, c_1 \geqq 0$ and a function $h \in L^{q'}(Q)\ (q' = q/q-1)$, such that

$$|A_\alpha(x, t, \eta, \xi)| \leqq c_1\{|\eta|^{q-1} + |\xi|^{q-1} + h(x, t)\}$$

for almost all $(x, t) \in Q, \forall\, |\alpha| \leqq m, \forall (\eta, \xi)$.

(A2)
$$\sum_{|\alpha|=m} (A_\alpha(x, t, \eta, \xi) - A_\alpha(x, t, \eta, \xi^*))(\xi_\alpha - \xi_\alpha^*) > 0$$

for almost all $(x, t) \in Q$, $\forall \eta$, $\forall \xi, \xi^*$ with $\xi \neq \xi^*$.

(A3) $\sum_{|\alpha|=m} A_\alpha(x, t, \eta, \xi)\xi_\alpha \geqq c_2 |\xi|^q$ for almost all $(x, t) \in Q$, $\forall (\eta, \xi)$, with some $c_2 > 0$.

(A4) There exists $z \in L^1(Q)$ with

$$\sum_{|\alpha| \leqq m-1} A_\alpha(x, t, \eta, \xi)\eta_\alpha \geqq z(x, t) \quad \text{for almost all} \quad (x, t) \in Q, \forall (\eta, \xi).$$

Without loss of generality we may assume $z(x, t) \leqq 0$ almost everywhere in Q.

Let $\mathcal{V}$ denote the space $L^q(0, T; V)$, with $V = W_0^{m,q}(\Omega)$. Because of (A1) the semilinear form

$$a(u, v) = \sum_{|\alpha| \leqq m} \int_Q A_\alpha(x, t, \delta u, D^m u) D^\alpha v \, dx \, dt$$

is defined on $\mathcal{V} \times \mathcal{V}$.

We set $\mathcal{W} = \mathcal{V} \cap L^2(Q)$ with the norm $\|\cdot\|_{\mathcal{W}} = \|\cdot\|_{\mathcal{V}} + \|\cdot\|_{L^2(Q)}$. $\mathcal{W}$ is a reflexive Banach space with dual $\mathcal{W}' = \mathcal{V}' + L^2(Q)$. Since

$$\mathcal{W} \subset L^2(Q) \subset \mathcal{W}' \subset L^1(0, T; V' + L^2(\Omega))$$

we may regard $u \in \mathcal{W}$ as a distribution on $(0, T)$ with values in $V' + L^2(\Omega)$. Hence its distributional derivative with respect to t exists, and so the condition $\partial u/\partial t \in \mathcal{W}'$ is meaningful.

When a norm $\|\cdot\|_X$ in a space X is considered, we will omit the subscript X if no confusion is possible. The same for duality pairings $(f, u)_X$ between two elements f in X' and u in X.

For any given $u_0 \in L^2(\Omega)$ let

$$\mathcal{U} = \left\{ u \mid u \in \mathcal{W}, \frac{\partial u}{\partial t} \in \mathcal{W}', u(0) = u_0 \right\}.$$

The condition $u(0) = u_0$ is justified by

LEMMA 1. *Let $u \in \mathcal{W}$ with $\partial u/\partial t \in \mathcal{W}'$. Then u is, eventually after modification on a set of measure zero in $[0, T]$, a continuous function on $[0, T]$ with values in $L^2(\Omega)$, and for u, v with these properties we have*

$$\left(\frac{\partial u}{\partial t}, v\right)_{\mathcal{W}} + \left(\frac{\partial v}{\partial t}, u\right)_{\mathcal{W}} = (u(T), v(T))_{L^2(\Omega)} - (u(0), v(0))_{L^2(\Omega)}.$$

The proof given in [**10**, p. 75–80] applies here with some obvious modifications. We have the following

THEOREM: *If*

$$\frac{a(u, u)}{\|u\|_{\mathcal{V}}} \to \infty \quad \text{for all } u \text{ in } \mathcal{V} \text{ with} \quad \|u\|_{\mathcal{V}} \to \infty \tag{1}$$

then given any f in $L^{q'}(Q)$ and u_0 in $L^2(\Omega)$ there exists a weak solution of (P), i.e. a u in $\mathcal{U}$ with

$$\left(\frac{\partial u}{\partial t}, w\right) + a(u, w) = (f, w) \quad \forall w \in \mathcal{W}.$$

Thus [**8**, Theorem 2.1] is essentially carried over to unbounded domains.

3. PROOF OF THE THEOREM

The proof is performed in several steps. First, a substitution leads us to a coercive problem in $\mathcal{W}$. For this problem the existence of a weak solution can be shown by approximating it in a similar way as in the elliptic case [**5**]. When the substitution is reversed, we get the statement of the theorem.

(A) With the substitution $u = e^{kt}v$ for some constant $k > 0$ we get formally from (P):

$$\begin{aligned} \frac{\partial v}{\partial t} + \sum_{|\alpha| \leq m} (-1)^{|\alpha|} e^{-kt} D^\alpha A_\alpha(x, t, e^{kt}\delta v, e^{kt} D^m v) + kv &= e^{-kt} f \quad \text{in } Q \\ v &= 0 \qquad\quad \text{on } \Sigma \\ v(0) &= u_0 \qquad \text{in } \Omega. \end{aligned}$$

Remark: This step is superfluous if $q = 2$, since then we already have $\mathcal{V} \subset L^2(Q)$. We introduce the following notations:

$$\tilde{f} = e^{-kt} f$$
$$\tilde{A}_\alpha(x, t, \eta, \xi) = e^{-kt} A_\alpha(x, t, e^{kt}\eta, e^{kt}\xi).$$

LEMMA 2. *The $\tilde{A}_\alpha$ satisfy the conditions* (A1) *to* (A4) (*with new constants in the inequalities*) *and the corresponding semilinear form $\tilde{a}(.,.)$ is coercive on $\mathcal{V}$.*

Proof. The calculations are simple. We only show (A3):

$$\sum_{|\alpha|=m} \tilde{A}_\alpha(x, t, \eta, \xi)\xi_\alpha = \sum_{|\alpha|=m} e^{-2kt} A_\alpha(x, t, e^{kt}\eta, e^{kt}\xi)(e^{kt}\xi_\alpha)$$
$$\geqq c_2 e^{-2kt} |e^{kt}\xi|^q = c_2 e^{kt(q-2)} |\xi|^q$$
$$\geqq \bar{c}_2 |\xi|^q \text{ almost everywhere in } Q, \quad \forall(\eta, \xi),$$

with

$$\bar{c}_2 = \begin{cases} c_2 & \text{for} \quad q \geqq 2 \\ c_2 e^{kT(q-2)} & \text{for} \quad 1 < q < 2. \end{cases}$$

Our modified problem becomes

$$(\mathrm{P}') \begin{cases} \dfrac{\partial v}{\partial t} + \displaystyle\sum_{|\alpha| \leqq m} (-1)^{|\alpha|} D^\alpha \tilde{A}_\alpha(x, t, \delta v, D^m v) + kv = \tilde{f} & \text{in } Q \\ v = 0 & \text{on } \Sigma \\ v(0) = u_0 & \text{in } \Omega, \end{cases}$$

with

$$\frac{\tilde{a}(v, v) + k \|v\|^2_{L^2(Q)}}{\|v\|_{\mathcal{W}}} \to \infty \quad \text{as} \quad \|v\|_{\mathcal{W}} \to \infty. \tag{2}$$

(B) For each n in $\mathbb{N}$ and u, v, w in $\mathcal{W}$ we define

$$\tilde{a}_{\omega_n}(u, v, w) = \tilde{a}_{1,\omega_n}(u, v, w) + \tilde{a}_{2,\omega_n}(u, w)$$
$$= \sum_{|\alpha|=m} \int_Q \tilde{A}_\alpha(x, t, \chi_{\omega_n}\delta u, D^m v) D^\alpha w \, dx \, dt$$
$$+ \sum_{|\alpha| \leqq m-1} \int_Q \chi_{\omega_n} \tilde{A}_\alpha(x, t, \delta u, D^m u) D^\alpha w \, dx \, dt,$$

$$b(v, w) = \sum_{|\alpha| \leqq m-1} \int_Q |D^\alpha v|^{q-2} D^\alpha v D^\alpha w \, dx \, dt,$$

where χ_{ω_n} denotes the characteristic function of the subdomain ω_n of Ω (independent of t). Let $\lambda > 0$ be fixed. As the linear form

$$w \mapsto c_n(u, v, w) = \tilde{a}_{\omega_n}(u, v, w) + \lambda b(v, w) + k(v, w)_{L^2(Q)}$$

is continuous on $\mathcal{W}$ for fixed u, v in $\mathcal{W}$, it induces a mapping $\mathcal{A}_n(.,.)$: $\mathcal{W} \times \mathcal{W} \to \mathcal{W}'$ through

$$c_n(u, v, w) = (\mathcal{A}_n(u, v), w).$$

Condition (A1) implies that this operator is bounded and continuous.

We set $\mathcal{A}_n(u, u) = \mathcal{A}_n(u)$. Then $\mathcal{A}_n(\cdot)$ is coercive, since

$$(\mathcal{A}_n(u), u) \geqq c_2 \|D^m u\|^q_{L^q(Q)} + \text{const} + \lambda \|u\|^q_{L^q(0, T; W^{m-1,q}(\Omega))} + k \|u\|^2_{L^2(Q)}$$
$$\geqq c' \|u\|^q_{\mathcal{V}} + c'' + k \|u\|^2_{L^2(Q)}, \tag{3}$$

with constants $c' > 0$, c'' that are independent of n.

LEMMA 3. *The operator $\mathscr{A}_n$ satisfies the following conditions:*

(i) $(\mathscr{A}_n(u,u)-\mathscr{A}_n(u,v),u-v)\geqq 0 \quad \forall u, v\in \mathscr{W}$.

(ii) *If $u_j \rightharpoonup u$ in $\mathscr{W}$, $\partial u_j/\partial t \rightharpoonup \partial u/\partial t$ in $\mathscr{W}'$ and $(\mathscr{A}_n(u_j,u_j)-\mathscr{A}_n(u_j,u),u_j-u)\to 0$, then $\mathscr{A}_n(u_j,v)\rightharpoonup \mathscr{A}_n(u,v)$ in $\mathscr{W}'$ for all v in $\mathscr{W}$.*

(iii) *Let $u_j \rightharpoonup u$ in $\mathscr{W}$, $\partial u_j/\partial t \rightharpoonup \partial u/\partial t$ in $\mathscr{W}'$ and $\mathscr{A}_n(u_j,v)\rightharpoonup \gamma$ in $\mathscr{W}'$ (for fixed v in $\mathscr{W}$). Then $(\mathscr{A}_n(u_j,v),u_j)\to(\gamma,u)$.*

Proof.

(i) is easily seen.

(ii) $$(\mathscr{A}_n(u_j,u_j)-\mathscr{A}_n(u_j,u),u_j-u)=\tilde a_{1,\omega_n}(u_j,u_j,u_j-u)-\tilde a_{1,\omega_n}(u_j,u,u_j-u)$$
$$+\lambda b(u_j,u_j-u)-\lambda b(u,u_j-u)+k(u_j-u,u_j-u)_{L^2(Q)}.$$

Since the last three parts in the sum together are positive, there follows

$$\limsup_{j\to\infty}\int_Q\sum_{|\alpha|=m}(\tilde A_\alpha(x,t,\chi_{\omega_n}\delta u_j,D^m u_j)\;-\tilde A_\alpha(x,t,\chi_{\omega_n}\delta u_j,D^m u))D^\alpha(u_j-u)\,dx\,dt\leqq 0.$$

We now use Aubin's lemma [**9**, p. 57]. It states, that

$$\mathscr{X}=\begin{cases}\left\{u \mid u\in L^q(0,T;W^{m,q}(\omega_n)),\dfrac{\partial u}{\partial t}\in L^{q'}(0,T;W^{-m,q'}(\omega_n))\right\} & \text{for} \quad q\geqq 2\\ \left\{u \mid u\in L^q(0,T;W^{m,q}(\omega_n)),\dfrac{\partial u}{\partial t}\in L^{q}(0,T;W^{-m,q}\;(\omega_n))\right\} & \text{for} \quad q<2\end{cases}$$

is compactly embedded in $L^q(0,T;W^{m-1,q}(\omega_n))$.

The same steps as in the proof of pseudomonotony for elliptic operators in bounded domains [**9**, p. 184] give us

$$D^\alpha u_j(x,t)\to D^\alpha u(x,t) \quad \text{almost everywhere on} \quad \omega_n\times(0,T), \quad \forall\,|\alpha|\leqq m,$$

and hence for fixed v in $\mathscr{W}$:

$$\tilde A_\alpha(.,.,\chi_{\omega_n}\delta u_j,D^m v)\to \tilde A_\alpha(.,.,\chi_{\omega_n}\delta u,D^m v)\text{ in }L^{q'}(Q), \quad \forall\,|\alpha|=m$$
$$\chi_{\omega_n}\tilde A_\alpha(.,.,\delta u_j,D^m u_j)\rightharpoonup \chi_{\omega_n}\tilde A_\alpha(.,.,\delta u,D^m u)\text{ in }L^{q'}(Q), \quad \forall\,|\alpha|\leqq m-1,$$

with which

$$(\mathscr{A}_n(u_j,v),w)=\tilde a_{1,\omega_n}(u_j,v,w)+\tilde a_{2,\omega_n}(u_j,w)+\lambda b(v,w)+k(v,w)_{L^2(Q)}$$
$$\to(\mathscr{A}_n(u,v),w) \quad \forall w\in\mathscr{W} \quad (j\to\infty).$$

(iii)

$$(\mathscr{A}_n(u_j,v),u_j)=\tilde a_{1,\omega_n}(u_j,v,u_j)+\tilde a_{2,\omega_n}(u_j,u_j)+\lambda b(v,u_j)+k(v,u_j)_{L^2(Q)}.$$
$$\tilde a_{2,\omega_n}(u_j,u)=(\mathscr{A}_n(u_j,v),u)-\tilde a_{1,\omega_n}(u_j,v,u)-\lambda b(v,u)-k(v,u)_{L^2(Q)}$$
$$\to(\gamma,u)-\tilde a_{1,\omega_n}(u,v,u)-\lambda b(v,u)-k(v,u)_{L^2(Q)} \quad (j\to\infty).$$

Applying Aubin's lemma we get

$$\lim_{j\to\infty}\tilde a_{2,\omega_n}(u_j,u_j-u)=0,$$

because $\chi_{\omega_n} D^\alpha(u_j - u) \to 0$ in $L^q(Q)$ $(j \to \infty)$ $\forall\, |\alpha| \leqq m-1$.

Furthermore

$$\tilde{a}_{1,\omega_n}(u_j, v, u_j) \to \tilde{a}_{1,\omega_n}(u, v, u) \quad (j \to \infty)$$

holds. And so

$$\begin{aligned}\lim_{j\to\infty} (\mathscr{A}_n(u_j, v), u_j) &= \lim_{j\to\infty} \{\tilde{a}_{1,\omega_n}(u_j, v, u_j) + \tilde{a}_{2,\omega_n}(u_j, u) + \lambda b(v, u_j) + k(v, u_j)\} \\ &\quad + \lim_{j\to\infty} \tilde{a}_{2,\omega_n}(u_j, u_j - u) \\ &= (\gamma, u).\end{aligned}$$

As Lemma 3 is now proved, we are in the position to refer to [**8** Theorem 2.1], i.e. for each $n \in \mathbb{N}$ there exists a $v_n \in \mathscr{U}$ with

$$\left(\frac{\partial v_n}{\partial t}, w\right) + (\mathscr{A}_n(v_n), w) = (\tilde{f}, w) \quad \forall w \in \mathscr{W}. \tag{4}$$

(C) Passage to the limit $n \to \infty$. First, we set $w = v_n$ in (4). With

$$\left(\frac{\partial v_n}{\partial t}, v_n\right) = \tfrac{1}{2}\{\|v_n(T)\|^2_{L^2(\Omega)} - \|v_n(0)\|^2_{L^2(\Omega)}\} \geqq \text{const}$$

and (3), there follows that

$$\|v_n\|_{\mathscr{W}} \leqq \text{const}.$$

Out of (4) we get

$$\left\|\frac{\partial v_n}{\partial t}\right\|_{\mathscr{W}'} \leqq \text{const},$$

and so we can extract a subsequence with

$$\begin{gathered} v_n \rightharpoonup v \quad \text{in} \quad \mathscr{W} \\ \frac{\partial v_n}{\partial t} \rightharpoonup \frac{\partial v}{\partial t} \quad \text{in} \quad \mathscr{W}' \quad (n \to \infty) \end{gathered}$$

for a $v \in \mathscr{U}$ (note that $\mathscr{U}$ is closed and convex and consequently also weakly closed).

For a fixed $i \in \mathbb{N}$ set $w = \varphi_i(v_n - v)$ in (4). This time Aubin's lemma gives

$$b(v_n, \varphi_i(v_n - v)) \to 0 \quad (n \to \infty).$$

Moreover

$$\begin{aligned}\liminf_{n\to\infty} &\left\{\left(\frac{\partial v_n}{\partial t}, \varphi_i(v_n - v)\right) + k(v_n, \varphi_i(v_n - v))_{L^2(Q)}\right\} \\ &= \liminf_{n\to\infty} \left\{\left(\frac{\partial v_n}{\partial t} - \frac{\partial v}{\partial t}, \varphi_i(v_n - v)\right) + k(v_n - v, \varphi_i(v_n - v))_{L^2(Q)}\right\} \\ &\geqq 0 + 0 = 0.\end{aligned}$$

The non-negativity of the first term will be shown in the appendix. Therefore we get out of (4)

$$\limsup_{n\to\infty} \tilde{a}_{\omega_n}(v_n, v_n, \varphi_i(v_n - v)) \leqq 0.$$

Similar considerations as in the proof of Lemma 3 lead to

$$D^\alpha v_n(x, t) \to D^\alpha v(x, t) \quad \text{almost everywhere on} \quad \omega_i \times (0, T), \quad \forall\, |\alpha| \leqq m$$

and hence, by a diagonal process, almost everywhere on Q, and

$$\tilde{a}_{\omega_n}(v_n, v_n, w) \to \tilde{a}(v, w) \quad \forall w \in \mathscr{W} \quad (n \to \infty).$$

Thus

$$\left(\frac{\partial v}{\partial t}, w\right) + \tilde{a}(v, w) + \lambda b(v, w) + k(v, w)_{L^2(Q)} = (\tilde{f}, w)_{L^q(Q)} \quad \forall w \in \mathscr{W}, \quad \text{with} \quad v \in \mathscr{U}.$$

The limit $\lambda \downarrow 0$ is analogous and implies the existence of a weak solution v of (P'), i.e. of $v \in \mathscr{U}$ with

$$\left(\frac{\partial v}{\partial t}, w\right) + \tilde{a}(v, w) + k(v, w)_{L^2(Q)} = (\tilde{f}, w)_{L^q(Q)} \quad \forall w \in \mathscr{W}. \tag{5}$$

(D) Finally, we must reverse the substitution $u = e^{kt}v$. We see that $u \in \mathscr{W}$ and

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} e^{kt} + ke^{kt}v \in \mathscr{W}'.$$

Equation (5) becomes

$$\left(e^{-kt}\frac{\partial u}{\partial t}, w\right) + a(u, e^{-kt}w) = (f, e^{-kt}w)_{L^q(Q)} \quad \forall w \in \mathscr{W}.$$

We obtain (compare the appendix)

$$\left(\frac{\partial u}{\partial t}, e^{-kt}w\right) + a(u, e^{-kt}w) = (f, e^{-kt}w),$$

from which we conclude

$$\left(\frac{\partial u}{\partial t}, w\right) + a(u, w) = (f, w) \quad \forall w \in \mathscr{W}.$$

Thus the Theorem is proved.

4. Appendix

(E) If $\psi \in C_0^\infty(\mathbb{R}^N)$ then we have $\psi u \in \mathscr{W}$ for all $u \in \mathscr{W}$. The distributional derivative of ψu with respect to t is therefore defined with values in $V' + L^2(\Omega)$.

Proposition. *For all $u \in \mathscr{W}$ with $\partial u/\partial t \in \mathscr{W}'$ the following holds:*

$$\frac{\partial}{\partial t}(\psi u) = \psi \frac{\partial u}{\partial t} \quad \text{in} \quad \mathscr{W}'. \tag{6}$$

Proof. For simplicity we write u' instead of $\partial u/\partial t$. First, we show that $\psi u' \in \mathcal{W}'$. For this purpose we split $u' = u_1 + u_2$ into its components $u_1 \in \mathcal{V}'$ and $u_2 \in L^2(Q)$, and get

$$(u', \psi w)_{\mathcal{W}} = (u_1, \psi w)_{\mathcal{V}} + (u_2, \psi w)_{L^2(Q)} \quad \forall w \in \mathcal{W}.$$

By definition

$$(u_1, \psi w)_{\mathcal{V}} = \int_0^T (u_1(t), \psi w(t))_V \, dt.$$

For almost all fixed t, $u_1(t) \in V'$ is the extension to V of a distribution [**1**, p. 50]. Hence $\psi u_1(t) \in V'$ and

$$(u_1(t), \psi w(t))_V = (\psi u_1(t), w(t))_V,$$

and so we obtain

$$(u', \psi w)_{\mathcal{W}} = (\psi u', w)_{\mathcal{W}} \quad \forall w \in \mathcal{W}. \tag{7}$$

Thus

$$\|\psi u'\|_{\mathcal{W}'} = \sup_{\substack{w \in \mathcal{W} \\ \|w\|_{\mathcal{W}} = 1}} |(\psi u', w)| = \sup_{\substack{w \in \mathcal{W} \\ \|w\|_{\mathcal{W}} = 1}} |(u', \psi w)| \leqq \text{const}\, \|u'\|_{\mathcal{W}'}. \tag{8}$$

We still have to verify that $(\psi u)' = \psi u'$ as a distribution on $(0, T)$ with values in $V' + L^2(\Omega)$. Let $a \in V \cap L^2(\Omega)$ and $\phi \in \mathcal{D}((0, T))$. In what follows we use $\langle .\,, . \rangle$ for $(.\,, .)_{V \cap L^2(\Omega)}$.

$$\begin{aligned}
\left\langle \int_0^T \psi u'(t)\phi(t)\, dt, a \right\rangle &= \int_0^T \langle \psi u'(t), a\rangle \phi(t)\, dt \overset{\text{comp. (7)}}{=} \int_0^T \langle u'(t), \psi a\rangle \phi(t)\, dt \\
&= \left\langle \int_0^T u'(t)\phi(t)\, dt, \psi a \right\rangle = \left\langle -\int_0^T u(t)\phi'(t)\, dt, \psi a \right\rangle \\
&= \left(-\int_0^T u(t)\phi'(t)\, dt, \psi a \right)_{L^2(\Omega)} = -\int_0^T (u(t), \psi a)_{L^2(\Omega)} \phi'(t)\, dt \\
&= -\int_0^T (\psi u(t), a)_{L^2(\Omega)} \phi'(t)\, dt \\
&= \left\langle -\int_0^T (\psi u(t))\phi'(t)\, dt, a \right\rangle.
\end{aligned} \tag{9}$$

Here we used the fact that $\langle v, w\rangle = (v, w)_{L^2(\Omega)}$ for v, w in $V \cap L^2(\Omega)$, because $V \cap L^2(\Omega) \subset L^2(\Omega) \subset V' + L^2(\Omega)$. We are allowed to interchange duality pairings and integration with respect to t, as can be seen in [**7**, p. 80]. Since (9) is valid for every a in $V \cap L^2(\Omega)$, (6) follows from (8) and (9). So the Proposition is proved.

(F) We have to show that

$$(w', \varphi_i w) \geqq 0 \quad \text{for} \quad w = u - v \quad \text{with} \quad u, v \in \mathcal{U}.$$

Use (6) and (7) to see that

$$\begin{aligned}
(w', \varphi_i w) = (\varphi_i w', w) = ((\varphi_i w)', w) &= \tfrac{1}{2}[(w', \varphi_i w) + ((\varphi_i w)', w)] \\
&= \tfrac{1}{2}[(w, \varphi_i w)_{L^2(\Omega)}(T) - (w, \varphi_i w)_{L^2(\Omega)}(0)].
\end{aligned}$$

As $w(0) = 0$, the desired result follows.

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